数理解析研究所講究録
第 803 巻 1992 年 48-64

A Short Course on b-Functions and Vanishing Cycles

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§0. Introduction.

In this article, we use the notation appearing in **[H]** freely, and a $\mathcal{D}$-module means a left $\mathcal{D}$-module. Let X be a complex manifold, f a holomorphic function on X, and $\mathcal{M}$ a regular holonomic system on X. By Riemann-Hilbert (RH) correspondence, $\mathrm{DR}(\mathcal{M})$ is a perverse sheaf. Hence its nearby cycle ${}^p\psi_f(\mathrm{DR}(\mathcal{M}))$ and vanishing cycle ${}^p\phi_f(\mathrm{DR}(\mathcal{M}))$ are perverse sheaves on $f^{-1}(0)$. If $f^{-1}(0)$ is a smooth hypersurface, again by RH correspondence there should be holonomic $\mathcal{D}_{f^{-1}(0)}$-modules $\mathcal{M}'$ and $\mathcal{M}''$ such that ${}^p\psi_f(\mathrm{DR}(\mathcal{M})) = \mathrm{DR}(\mathcal{M}')$ and ${}^p\phi_f(\mathrm{DR}(\mathcal{M})) = \mathrm{DR}(\mathcal{M}'')$. Malgrange **[Ma]** and Kashiwara **[Kv]** have given such $\mathcal{M}'$ and $\mathcal{M}''$ by using the notion of V-filtration. When $f^{-1}(0)$ is not smooth, the situation is reduced to the smooth case by the graph map of f. There are already excellent surveys **[MS]**, **[S]** of this topic. This article may be considered as a very short version of **[MS]** or **[S]**. Although most proofs of assertions are omitted, those of Proposition 4.2 and 4.4 are exposed in order to convince readers that morphisms T, *can*, and *var* correspond to the counterparts mentioned there.

In §1 we define b-functions and look at some examples. In §2 we define V-filtrations, which can be calculated by b-functions. We also look at some examples again. In §3 we state the stability under standard operations of the category of coherent clD-modules which admit the canonical V-filtrations. In §4 we define moderate nearby cycles and moderate vanishing cycles, which turn out to be quasi-isomorphic to certain graded pieces of the canonical V-filtration. In §5 we recall nearby cycles and vanishing cycles, and state the main theorem (Theorem 5.1).

§1. b-Functions.

Let X be a complex manifold and f a holomorphic function on it. We set $\mathcal{D}_X[s] := \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$ where s is an indeterminate central element. Let $\mathcal{I}_f$ denote the left ideal of $\mathcal{D}_X[s]$ consisting of all operators $P(s,x,D)$ in $\mathcal{D}_X[s]$ such that $P(s,x,D)f(x)^s = 0$ holds for a generic x. A $\mathcal{D}_X[s]$-module $\mathcal{N}_f := \mathcal{D}_X[s]/\mathcal{I}_f$ has a $\mathcal{D}_X$-linear endomorphism t defined by $P(s)f^s \mapsto P(s+1)f^{s+1}$. Since we have $[t,s] = t$, $\mathcal{M}_f := \mathcal{N}_f/t\mathcal{N}_f$ is a $\mathcal{D}_X[s]$-module.

Definition 1.1 **[SSM]**, **[Be]**: The minimal polynomial $b(s)$ of the multiplication by s on $\mathcal{M}_f$ is said to be the b-function of f.

Theorem 1.2 **[Be]**, **[Bj]**, **[Kb]**. *The $\mathcal{D}_X$-module $\mathcal{M}_f$ is holonomic and the* *b-function of f locally exists.*

EXAMPLE 1.3 [**Mi**], [**Y**]: Let $X = \mathbb{C}^n$, $x_1, \dots, x_n$ a coordinate system on X and $D_i = \frac{\partial}{\partial x_i}$ $(1 \le i \le n)$. We assume f to have an isolated singularity at the origin and $f(0) = 0$. We suppose that there exist $v = \sum_{i=1}^{n} \frac{r_i}{r} x_i D_i$, $r \in \mathbb{Z}_{>0}$, $r_1, \dots, r_n \in \mathbb{Z}_{\geq 0}$ such that $v(f) = f$. The b-function of f at a point where df does not vanish is $s + 1$. Hence $s + 1$ is also a factor of the b-function $b(s)$ of f at the origin. Since $vf^s = sf^s$, $\mathcal{M}_f$ is a singly generated $\mathcal{D}_X$-module. Let $\bar{\mathcal{M}}_f = (s+1)\mathcal{M}_f$ and $\bar{b}(s)$ denote the minimal polynomial of s on $\bar{\mathcal{M}}_f$. Then we see that $b(s) = (s+1)\bar{b}(s)$ and $\mathcal{M}_f = \mathcal{D}_X/\mathcal{D}_X f_1 + \cdots + \mathcal{D}_X f_n$ where $f_i = D_i(f)$. Let v^* be the adjoint operator of v, i.e., $v^* = -\sum_{i=1}^{n} \frac{r_i}{r}(x_i D_i + 1)$. Then we see $\bar{b}(s) =$ the minimal polynomial of s on $\bar{\mathcal{M}}_f =$ the minimal polynomial of v on $\bar{\mathcal{M}}_f =$ the minimal polynomial of v^* on $\mathcal{O}_X/(f_1, \dots, f_n)$. For a monomial x^α where α is a multi-index, we have $v^*(x^\alpha) = -\sum_{i=1}^{n} \frac{r_i}{r}(\alpha_i + 1)x^\alpha$. We define a set R by $R = \{\sum_{i=1}^{n} \frac{r_i}{r}(\alpha_i + 1) \,|\, \{x^\alpha\}_\alpha \text{ is a basis for } \mathcal{O}_X/(f_1, \dots, f_n)\}$. Then we obtain $b(s) = (s+1)\prod_{\beta \in R}(s + \beta)$.

EXAMPLE 1.4: Let $X = \mathbb{C}^n$ and $f = x_1^{e_1} \cdots x_n^{e_n}$ where $e_i \in \mathbb{Z}_{\geq 0}$ $(1 \le i \le n)$. It is easy to check $D_1^{e_1} \cdots D_n^{e_n} f^{s+1} = \prod_{i=1}^{n} \prod_{k=1}^{e_i}(e_i s + k) f^s$. On the other hand we suppose that there exist an operator $P(s) \in \mathcal{D}_X[\,s\,]$ and a nonzero polynomial $b'(s) \in \mathbb{C}[s]$ such that $P(s)f^{s+1} = b'(s)f^s$. By the relative invariance under the action of $(\mathbb{C}^\times)^n$, it is easy to see that there exists $Q(s) \in \mathbb{C}[x_1D_1, \ldots, x_nD_n, s]$ such that $P(s) = Q(s)D_1^{e_1} \cdots D_n^{e_n}$. Therefore we see that the b-function of f at the origin is $\prod_{i=1}^n \prod_{k=1}^{e_i}(s + \frac{k}{e_i})$.

There are many other examples of b-functions which can be calculated. See **[Y]**, for instance, and **[SKKO]** for b-functions of relative invariants of prehomogeneous spaces. More generally Kashiwara has proved in **[K2]** that for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and a section $u \in \mathcal{M}$ there exists locally an operator $P(s) \in \mathcal{D}_X[s]$ and a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $P(s)f^{s+1}u = b(s)f^s u$. As an application, the holonomicity of $\mathcal{H}^i_{[X|f^{-1}(0)]}(\mathcal{M})$ has been proved there.

§2. V-Filtration.

First of all we introduce the lexicographical order in $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\sqrt{-1}$. Let Y be a smooth closed submanifold of X of codimension one, $\mathcal{I}_Y$ the defining ideal of Y. For $k \in \mathbb{Z}$ we define

$$V_k\mathcal{D}_X := \{\, P \in \mathcal{D}_X \mid P\mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k} \quad (\forall j \in \mathbb{Z})\,\}$$

where $\mathcal{I}_Y^j = \mathcal{O}_X$ for $j \leq 0$. Then $\{\, V_k\mathcal{D}_X \,\}_{k\in\mathbb{Z}}$ is an exhaustive increasing filtration. Let t be a local equation of Y and D_t a local vector field such that $[D_t, t] = 1$. We have $t \in V_{-1}\mathcal{D}_X$, $D_t \in V_1\mathcal{D}_X$, $\mathrm{gr}_0^V \mathcal{D}_X := V_0\mathcal{D}_X/V_{-1}\mathcal{D}_X = \mathcal{D}_Y[tD_t]$ and $V_k\mathcal{D}_X = \{\sum_{k\geq j-i} a_{ij}(y, D_y)t^iD_t^j\}$.

DEFINITION 2.1 **[Kv]**, **[Ma]**: Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. An increasing filtration $\{V_\alpha\mathcal{M}\}_{\alpha\in\mathbb{C}}$ satisfying the following conditions is called the canonical V-filtration.

(1) $\mathcal{M} = \cup_{\alpha\in\mathbb{C}} V_\alpha\mathcal{M}$. Each $V_\alpha\mathcal{M}$ is a coherent $V_0\mathcal{D}_X$-submodule.

(2) $(V_i\mathcal{D}_X)(V_\alpha\mathcal{M}) \subset V_{\alpha+i}\mathcal{M}$ $(\forall\alpha\in\mathbb{C}, \forall i \in \mathbb{Z})$.

(3) $t(V_\alpha\mathcal{M}) = V_{\alpha-1}\mathcal{M}$ $(\forall\alpha < 0)$.

(4) The action of $(tD_t + 1 + \alpha)$ on $\mathrm{gr}_\alpha^V \mathcal{M}$ $(\forall\alpha \in \mathbb{C})$ is nilpotent where $\mathrm{gr}_\alpha^V \mathcal{M} = V_\alpha\mathcal{M}/V_{<\alpha}\mathcal{M}$ and $V_{<\alpha}\mathcal{M} = \cup_{\beta<\alpha} V_\beta\mathcal{M}$.

REMARKS 2.2: (1) The definition of the canonical V-filtration does not depend on the choice of t and D_t. The canonical V-filtration is unique if it exists.

(2) Since the adjoint of $(tD_t + 1 + \alpha)$ is $-(tD_t - \alpha)$, the eigenvalue of tD_t on $\mathrm{gr}_\alpha^V \mathcal{N}$ is α for a right $\mathcal{D}_X$-module $\mathcal{N}$.

(3) $t : \mathrm{gr}_\alpha^V \mathcal{M} \longrightarrow \mathrm{gr}_{\alpha-1}^V \mathcal{M}$ and $D_t : \mathrm{gr}_{\alpha-1}^V \mathcal{M} \longrightarrow \mathrm{gr}_\alpha^V \mathcal{M}$ are bijective for

$\alpha \neq 0$.

DEFINITION 2.3: We say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is specializable along Y and we denote $\mathcal{M} \in B_Y$ if the following equivalent conditions are satisfied:

(1) For any system of local generators $u_1, \ldots, u_l$ of $\mathcal{M}$ there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $b(tD_t)u_i \in \sum_{j=1}^{l}(V_{-1}\mathcal{D}_X)u_j$ $(1 \leq \forall i \leq l)$.

(2) $\mathcal{M}$ admits the canonical V-filtration with respect to Y and there exists a finite set $A \subset \mathbb{C}$ such that $\{\alpha \in \mathbb{C} \mid \mathrm{gr}_\alpha^V \mathcal{M} \neq 0\} \subset A + \mathbb{Z}$.

Let $\mathcal{M} \in B_Y$ and $u \in \mathcal{M}$. Then there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $b(tD_t)u \in (V_{-1}\mathcal{D}_X)u$. The minimal polynomial among such is called the b-function of the section u. The canonical V-filtration of $\mathcal{M}$ is known to be given by $V_\alpha\mathcal{M} = \{u \in \mathcal{M} \mid$ all roots of the b-function of u are greater than or equal to $-\alpha - 1\}$.

PROPOSITION 2.4. *Let $0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$ be an exact sequence of coherent $\mathcal{D}_X$-modules. Then we have*

(1) $\mathcal{M} \in B_Y \iff \mathcal{M}', \mathcal{M}'' \in B_Y$.

(2) *The induced sequence $0 \longrightarrow V_\alpha\mathcal{M}' \longrightarrow V_\alpha\mathcal{M} \longrightarrow V_\alpha\mathcal{M}'' \longrightarrow 0$ is exact for*

$\forall\alpha \in \mathbb{C}$ *if* $\mathcal{M} \in B_Y$.

(3) *The induced sequence* $0 \to \mathrm{gr}^V_\alpha \mathcal{M}' \to \mathrm{gr}^V_\alpha \mathcal{M} \to \mathrm{gr}^V_\alpha \mathcal{M}'' \to 0$ *is exact for* $\forall\alpha \in \mathbb{C}$ *if* $\mathcal{M} \in B_Y$.

REMARK 2.5: Let $\mathcal{M} \in B_Y$. Then $\mathrm{gr}^V_\alpha \mathcal{M}$ is a coherent $\mathrm{gr}^V_0 \mathcal{D}_X = \mathcal{D}_Y[tD_t]$-module for any $\alpha \in \mathbb{C}$. Since the action of $(tD_t+1+\alpha)$ is nilpotent on $\mathrm{gr}^V_\alpha \mathcal{M}$, it is a coherent $\mathcal{D}_Y$-module.

EXAMPLE 2.6: Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with $\mathrm{Supp}(\mathcal{M}) \subset Y$, and $u \in \mathcal{M}$. Then there exists $i \in \mathbb{Z}_{>0}$ such that $t^i u = 0$. So we have $\prod_{k=1}^{i}(tD_t + k)u = D_t^i t^i u = 0$. Hence we obtain $\mathcal{M} = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathcal{M}_i$ where $\mathcal{M}_i = \{u \in \mathcal{M} \mid (tD_t + 1 + i)u = 0\}$, and $V_\alpha \mathcal{M} = \bigoplus_{i \leq \alpha} \mathcal{M}_i$.

EXAMPLE 2.7: Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We assume Y to be non-characteristic for $\mathcal{M}$, i.e., $\mathrm{Ch}(\mathcal{M}) \cap T^*_Y X \subset T^*_X X$. Then $\mathcal{M} \in B_Y$. The proof could be reduced to the case of $\mathcal{D}_X v = \mathcal{D}_X / \mathcal{D}_X P$ with $P \in V_N \mathcal{D}_X$, $\bar{P} = \bar{D}_t^N \in V_N \mathcal{D}_X / V_{N-1} \mathcal{D}_X$ and $N \in \mathbb{Z}_{>0}$ where the bar indicates the canonical image. Since $Pv = 0$, in $(V_N \mathcal{D}_X)v / (V_{N-1}\mathcal{D}_X)v$ we have $\bar{P}\bar{v} = \bar{D}_t^N \bar{v} = 0$, i.e., $D_t^N v \in V_{N-1}\mathcal{D}_X v$. Hence we obtain $\prod_{k=0}^{N-1}(tD_t - k)v = t^N D_t^N v \in V_{-1}\mathcal{D}_X v$. In general, any root of the b-function of any section of $\mathcal{M}$ is a nonnegative integer. Therefore we see by Definition 2.1 (3)

$$V_\alpha \mathcal{M} = \begin{cases} t^{-[\alpha]-1}\mathcal{M} & \alpha < -1 \\ \\ \mathcal{M} & \alpha \geq -1 \end{cases}$$

where $[\alpha] = \max\{\, n \in \mathbb{Z} \,|\, n \leq \alpha \,\}$.

Let f be a holomorphic function, $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module, $u \in \mathcal{M}$ and $Y = f^{-1}(0)$. Let i_f denote the graph of $f : X \to X \times \mathbb{C}$ and t the coordinate of $\mathbb{C}$ in $X \times \mathbb{C}$. Then we see $\mathcal{M} \in B_Y \Leftrightarrow i_{f*}\mathcal{M} \in B_{X\times\{0\}}$. Furthermore there is the following correspondence under the isomorphism of $\mathcal{D}_X[\,s,t\,]$ onto $V_0(\mathcal{D}_X[\,t, D_t\,]) = \mathcal{D}_X[\,t, tD_t\,]$:

$$s \longleftrightarrow -D_t t$$

$$\mathcal{D}_X[\,s\,]f^s u \longleftrightarrow V_0(\mathcal{D}_X[\,t, D_t\,])u \otimes \delta(t-f)$$

$$P(s)f^{s+1}u = b(s)f^s u \longleftrightarrow P(-D_t t)tu \otimes \delta(t-f) = b(-D_t t)u \otimes \delta(t-f).$$

By Kashiwara's result recalled in §1, we obtain:

PROPOSITION 2.8. *All holonomic $\mathcal{D}_X$-modules belong to B_Y.*

§3. Operations in B_Y.

PROPOSITION 3.1. *Let $Y = \{\, t = 0 \,\}$ and $\mathcal{M} \in B_Y$. Then*

(1) $\mathcal{M}[t^{-1}] \in B_Y$.

(2) $\mathcal{H}^j(\mathcal{M}^*) \in B_Y$ *for* $\forall j$. *Moreover for* $\forall j$ *locally we have isomorphisms* $\mathrm{gr}^V_\alpha(\mathcal{H}^j(\mathcal{M}^*)) \xrightarrow{\sim} \mathcal{H}^j(\mathrm{gr}^V_{-\alpha-1}(\mathcal{M})^*)$ $(-1 < \alpha < 0)$ *and* $\mathrm{gr}^V_\beta(\mathcal{H}^j(\mathcal{M}^*)) \xrightarrow{\sim} \mathcal{H}^j(\mathrm{gr}^V_\beta(\mathcal{M})^*)$ $(\beta = -1, 0)$. *Under these isomorphisms the transpose* ${}^t t : \mathcal{H}^j(\mathrm{gr}^V_{-1}(\mathcal{M})^*) \to \mathcal{H}^j(\mathrm{gr}^V_0(\mathcal{M})^*)$ *corresponds to* $-D_t : \mathrm{gr}^V_{-1}(\mathcal{H}^j(\mathcal{M}^*)) \to \mathrm{gr}^V_0(\mathcal{H}^j(\mathcal{M}^*))$ *and the transpose* ${}^t D_t : \mathcal{H}^j(\mathrm{gr}^V_0(\mathcal{M})^*) \to \mathcal{H}^j(\mathrm{gr}^V_{-1}(\mathcal{M})^*)$ *corresponds to* $t : \mathrm{gr}^V_0(\mathcal{H}^j(\mathcal{M}^*)) \to \mathrm{gr}^V_{-1}(\mathcal{H}^j(\mathcal{M}^*))$.

PROPOSITION 3.2. *Let* $\mathcal{M}$ *be a holonomic* $\mathcal{D}_X$*-module and* i *the inclusion of* Y *into* X. *Then*

(1) *The restriction* $i^*\mathcal{M}$ *is quasi-isomorphic to* $0 \to \mathrm{gr}^V_{-1}\mathcal{M} \xrightarrow{D_t} \underset{\bullet}{\mathrm{gr}^V_0\mathcal{M}} \to 0$ *where the dot indicates the place of degree zero.*

(2) *For any* $\alpha \in \mathbb{C}$, $\mathrm{gr}^V_\alpha\mathcal{M}$ *is a holonomic* $\mathcal{D}_Y$*-module.*

PROOF: (1) By Remark 2.2 (3) and Proposition 3.1 (2) we have

$$\begin{aligned} i^!\mathcal{M}^* &\underset{qis}{\xrightarrow{\sim}} (0 \to \underset{\bullet}{\mathrm{gr}^V_0(\mathcal{M}^*)} \xrightarrow{t} \mathrm{gr}^V_{-1}(\mathcal{M}^*) \to 0) \\ &\underset{qis}{\xrightarrow{\sim}} (0 \to \underset{\bullet}{(\mathrm{gr}^V_0\mathcal{M})^*} \xrightarrow{{}^tD_t} (\mathrm{gr}^V_{-1}\mathcal{M})^* \to 0). \end{aligned}$$

Since $i^*\mathcal{M} = (i^!\mathcal{M}^*)^*$, we obtain the assertion.

(2) We know that

$$\mathcal{M} : \text{holonomic} \Leftrightarrow \mathcal{H}^j(\mathcal{M}^*) = 0 \quad \text{for } j \neq 0.$$

Hence by Proposition 3.1 (2) we obtain $\mathcal{H}^j((\mathrm{gr}^V_\alpha \mathcal{M})^*) = 0$ for $j \neq 0$. This means the holonomicity of $\mathrm{gr}^V_\alpha \mathcal{M}$.

PROPOSITION 3.3. *Let $g : X' \to X$ be a proper morphism of smooth manifolds. We suppose that $Y' := g^{-1}(Y)$ is a smooth hypersurface and $\mathcal{M} \in B_{Y'}$ has a global good filtration. Then for any j, we have $\mathcal{H}^j(\mathbb{R}g_*\mathcal{M}) \in B_Y$ and the canonical V-filtration of $\mathcal{M}$ induces the one for $\mathcal{H}^j(\mathbb{R}g_*\mathcal{M})$.*

§4. Moderate Nearby Cycles and Moderate Vanishing Cycles.

Let Y be a smooth hypersurface defined by $t : X \to \mathbb{C}$. For a coherent $\mathcal{D}_X$-module $\mathcal{M} \in B_Y$, $p \in \mathbb{Z}_{\geq 0}$ and $-1 \leq \alpha < 0$, we define

$$\mathcal{M}_{\alpha,p} := \bigoplus_{0 \leq k \leq p} \mathcal{M}[t^{-1}] \otimes e_{\alpha,k}$$

where $e_{\alpha,k} = t^{\alpha+1}(\mathrm{Log}\, t)^k/k!$. It is clear that for any $\beta \in \mathbb{C}$

$$V_\beta \mathcal{M}_{\alpha,p} = \bigoplus_{0 \leq k \leq p} V_{\beta+\alpha+1}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k}.$$

Then the monodromy $T = \exp(2\pi i t D_t)$ induces a $\mathcal{D}_Y$-automorphism on $\mathcal{M}_{\alpha,p}$ by $T(m \otimes e_{\alpha,k}) = m \otimes T(e_{\alpha,k})$, and accordingly on $\mathrm{gr}^V_\beta(\mathcal{M}_{\alpha,p})$.

DEFINITION 4.1: For $-1 \leq \alpha \leq 0$, we define the moderate nearby cycle $\psi^m_{t,\alpha}(\mathcal{M})$ by

$$\psi^m_{t,\alpha}(\mathcal{M}) := \varinjlim_p \psi^m_{t,\alpha,p}(\mathcal{M})$$

where $\psi^m_{t,\alpha,p}(\mathcal{M}) := i^*(\mathcal{M}_{\alpha,p})[-1]$.

By Proposition 3.2 we see

$$\psi^m_{t,\alpha,p}(\mathcal{M}) \xrightarrow[qis]{\sim} (0 \to \underset{\bullet}{\mathrm{gr}^V_{-1}(\mathcal{M}_{\alpha,p})} \xrightarrow{D_t} \mathrm{gr}^V_0(\mathcal{M}_{\alpha,p}) \to 0).$$

We remark that T acts on $\psi^m_{t,\alpha}(\mathcal{M})$ as well.

PROPOSITION 4.2. *For $\mathcal{M} \in B_Y$ and $-1 \leq \alpha < 0$, we have a quasi-isomorphism $\mathrm{gr}^V_\alpha \mathcal{M} \xrightarrow[qis]{\sim} \psi^m_{t,\alpha}(\mathcal{M})$. Here the action of T on $\psi^m_{t,\alpha}(\mathcal{M})$ corresponds to that of* $\exp(-2\pi i t D_t)$.

PROOF: Since $V_{<0}\mathcal{M} = V_{<0}(\mathcal{M}[t^{-1}])$, we have

$$\mathrm{gr}^V_{-1}(\mathcal{M}_{\alpha,p}) = \bigoplus_{0 \leq k \leq p} \mathrm{gr}^V_\alpha(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k} = \bigoplus_{0 \leq k \leq p} \mathrm{gr}^V_\alpha(\mathcal{M}) \otimes e_{\alpha,k}.$$

As $\mathcal{M}_{\alpha,p} = \mathcal{M}_{\alpha,p}[t^{-1}]$, we know $\mathcal{H}^0(\psi^m_{t,\alpha,p}(\mathcal{M})) = \mathrm{Ker}(D_t) = \mathrm{Ker}\ (tD_t : \mathrm{gr}^V_{-1}(\mathcal{M}_{\alpha,p}) \to \mathrm{gr}^V_{-1}(\mathcal{M}_{\alpha,p}))$. Since $tD_t(m \otimes e_{\alpha,k}) = [(tD_t + \alpha + 1)m] \otimes$

$e_{\alpha,k} + m \otimes e_{\alpha,k-1}$, we see $\sum_{k=0}^{p} m_k \otimes e_{\alpha,k} \in \mathrm{Ker}(tD_t) = \mathcal{H}^0(\psi^m_{t,\alpha,p}(\mathcal{M})) \Leftrightarrow (tD_t+\alpha+1)m_k+m_{k+1} = 0 \ (0 \le \forall k \le p-1) \Leftrightarrow m_k = [-(tD_t+\alpha+1)]^k m_0 \ (0 \le \forall k \le p)$. Hence for p such that $(tD_t+\alpha+1)^p = 0$ on $\mathrm{gr}^V_\alpha(\mathcal{M})$, the morphism $\mathrm{gr}^V_\alpha(\mathcal{M}) \ni m_0 \mapsto \sum_{k=0}^{p}[-(tD_t+\alpha+1)]^k m_0 \otimes e_{\alpha,k} \in \mathcal{H}^0(\psi^m_{t,\alpha,p}(\mathcal{M}))$ is isomorphic.

Let $x = \sum_{k=0}^{p}[-(tD_t+\alpha+1)]^k m_0 \otimes e_{\alpha,k} \in \mathrm{Ker}(tD_t)$. Then we have $0 = (tD_t)x = \sum_{k=0}^{p}[-(tD_t+\alpha+1)]^k (tD_t) m_0 \otimes e_{\alpha,k} + \sum_{k=0}^{p}[-(tD_t+\alpha+1)]^k m_0 \otimes (tD_t) e_{\alpha,k}$, and thus $\sum_{k=0}^{p}[-(tD_t+\alpha+1)]^k m_0 \otimes (2\pi i tD_t) e_{\alpha,k} = \sum_{k=0}^{p}[-(tD_t+\alpha+1)]^k ((-2\pi i tD_t) m_0) \otimes e_{\alpha,k}$. Hence the monodromy T corresponds to $\exp(-2\pi i tD_t)$.

Since t induces an isomorphism $\mathrm{gr}^V_0(\mathcal{M}_{\alpha,p}) \xrightarrow{\sim} \mathrm{gr}^V_{-1}(\mathcal{M}_{\alpha,p})$, we see $\mathcal{H}^1(\psi^m_{t,\alpha,p}(\mathcal{M})) = \mathrm{Coker}(D_t) = \mathrm{Coker}(D_t t : \mathrm{gr}^V_0(\mathcal{M}_{\alpha,p}) \to \mathrm{gr}^V_0(\mathcal{M}_{\alpha,p}))$. For $\sum_{k=0}^{p} m_k \otimes e_{\alpha,k} \in \bigoplus_{0\le k\le p} \mathrm{gr}^V_{\alpha+1}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k} = \mathrm{gr}^V_0(\mathcal{M}_{\alpha,p})$, we have $D_t t(\sum_{k=0}^{p} m_k \otimes e_{\alpha,k}) = \sum_{k=0}^{p}((D_t t+\alpha+1)m_k \otimes e_{\alpha,k} + m_k \otimes e_{\alpha,k-1}) = \sum_{k=0}^{p} m'_k \otimes e_{\alpha,k}$ where $m'_k = (D_t t+\alpha+1)m_k + m_{k+1}$. Hence for l such that $(D_t t+\alpha+1)^l = 0$ on $\mathrm{gr}^V_{\alpha+1}(\mathcal{M}[t^{-1}])$, we have $m \otimes e_{\alpha,k} = D_t t(\sum_{i=1}^{l}[-(D_t t+\alpha+1)]^{i-1} m \otimes e_{\alpha,k+i})$ and thus $\mathcal{H}^1(\psi^m_{t,\alpha}(\mathcal{M})) = 0$.

Definition 4.3: We define the moderate vanishing cycle $\phi^m_{t,0}(\mathcal{M})$ to be the inductive limit of the mapping cone $\phi^m_{t,0,p}(\mathcal{M})$ of the natural morphism

$i^*\mathcal{M}[-1] \longrightarrow i^*\mathcal{M}_{-1,p}[-1] = \psi^m_{t,-1,p}(\mathcal{M})$, i.e.,

$$\phi^m_{t,0,p} = (0 \longrightarrow \operatorname{gr}^V_{-1}\mathcal{M} \xrightarrow{j\oplus -D_t} \underset{\bullet}{\operatorname{gr}^V_{-1}\mathcal{M}_{-1,p} \oplus \operatorname{gr}^V_0 \mathcal{M}} \xrightarrow{D_t+j} \operatorname{gr}^V_0 \mathcal{M}_{-1,p} \longrightarrow 0)$$

where j is the natural morphism $\mathcal{M} \longrightarrow \mathcal{M}_{-1,p} = \bigoplus_{0\le k\le p} \mathcal{M}[t^{-1}] \otimes e_{-1,k}$.

We define morphisms *can* : $\psi^m_{t,-1}(\mathcal{M}) \longrightarrow \phi^m_{t,0}(\mathcal{M})$ and *var* : $\phi^m_{t,0}(\mathcal{M}) \longrightarrow \psi^m_{t,-1}(\mathcal{M})$ by the morphisms $id : \psi^m_{t,-1,p}(\mathcal{M}) \longrightarrow \phi^m_{t,0,p}(\mathcal{M})$ and $T - id$: $\phi^m_{t,0,p}(\mathcal{M}) \longrightarrow \psi^m_{t,-1,p}(\mathcal{M})$ respectively.

PROPOSITION 4.4. *For $\mathcal{M} \in B_Y$, we have a quasi-isomorphism $\operatorname{gr}^V_0 \mathcal{M} \underset{qis}{\xrightarrow{\sim}} \phi^m_{t,0}(\mathcal{M})$. Moreover can corresponds to $D_t : \operatorname{gr}^V_{-1}\mathcal{M} \longrightarrow \operatorname{gr}^V_0\mathcal{M}$ and var to $[\frac{(\exp(-2\pi i t D_t)-1)}{tD_t}]t : \operatorname{gr}^V_0 \mathcal{M} \longrightarrow \operatorname{gr}^V_{-1}\mathcal{M}$.*

PROOF: Let $x = \sum_{k=0}^p m_k \otimes e_{-1,k} + n_0 \in \operatorname{gr}^V_{-1}\mathcal{M}_{-1,p} \oplus \operatorname{gr}^V_0 \mathcal{M} = (\bigoplus_{k=0}^p \operatorname{gr}^V_{-1}\mathcal{M} \otimes e_{-1,k}) \oplus \operatorname{gr}^V_0\mathcal{M}$. Then we can check

$$x \in \operatorname{Ker}(D_t + j) \Leftrightarrow (*) \quad \begin{cases} m_1 = -tD_t m_0 - tn_0 \\ m_{k+1} = -tD_t m_k \quad (k \ge 1). \end{cases}$$

Hence we obtain an isomorphism $\operatorname{gr}^V_{-1}\mathcal{M} \oplus \operatorname{gr}^V_0 \mathcal{M} \xrightarrow{\sim} \operatorname{Ker}(D_t + j)$ defined by $m_0 + n_0 \mapsto \sum m_k \otimes e_{-1,k} + n_0$ with (*). So we see $\operatorname{gr}^V_0\mathcal{M} \xrightarrow{\sim} \mathcal{H}^0(\phi^m_{t,0})$. Since $m \equiv D_t m \bmod \operatorname{Im}(j \oplus -D_t)$ for $m \in \operatorname{gr}^V_{-1}\mathcal{M}$, the morphism *can* corresponds to $D_t : \operatorname{gr}^V_{-1}\mathcal{M} \longrightarrow \operatorname{gr}^V_0 \mathcal{M}$. The element $\sum_{k\ge1}(-tD_t)^{k-1}(-tn) \otimes e_{-1,k} +$

$n \in \mathrm{Ker}(D_t + j)$ corresponds to $n \in \mathrm{gr}_0^V \mathcal{M}$. Since the coefficient of $(T - id)(\sum_{k\geq 1}(-tD_t)^{k-1}(-tn) \otimes e_{-1,k})$ at $e_{-1,0}$ is $\sum_{k\geq 1}(2\pi i)^k \frac{(-tD_t)^{k-1}}{k!}(-tn)$, the morphism *var* corresponds to $[\frac{(\exp(-2\pi i tD_t)-1)}{tD_t}]t : \mathrm{gr}_0^V \mathcal{M} \to \mathrm{gr}_{-1}^V \mathcal{M}$.

§5. Nearby Cycles and Vanishing Cycles.

Let f be a nonconstant holomorphic function on X, i the inclusion of $f^{-1}(0)$ into X and $K \in D_c^b(\mathbb{C}_X)$. Let $\tilde{\mathbb{C}}^\times$ denote the universal covering of $\mathbb{C}^\times$ and p the natural map $\tilde{X}^\times := X \times_{\mathbb{C}} \tilde{\mathbb{C}}^\times \to X$. Following **[SGA7]** we define the nearby cycle $\psi_f(K)$ by $\psi_f(K) := i^{-1}\mathbb{R}p_*p^{-1}K \in D_c^b(\mathbb{C}_{f^{-1}(0)})$. The natural morphism $K \longrightarrow \mathbb{R}p_*p^{-1}K$ induces a morphism $i^{-1}K \longrightarrow \psi_f(K)$, whose mapping cone $\phi_f(K) \in D_c^b(\mathbb{C}_{f^{-1}(0)})$ is called the vanishing cycle. By the definition of $\phi_f(K)$ we have the canonical morphism $can : \psi_f(K) \longrightarrow \phi_f(K)$. Associated to the canonical generator of $\pi_1(\mathbb{C}^\times)$ the monodromy automorphism T acts on $\psi_f(K)$ and $\phi_f(K)$. Since $(T - id)_{|i^{-1}K} = 0$, $T - id$ induces the variation $var : \phi_f(K) \longrightarrow \psi_f(K)$. For $\lambda \in \mathbb{C}^\times$ we define a subcomplex $\psi_{f,\lambda}(K)$ of $\psi_f(K)$ by

$$\psi_{f,\lambda}(K) := \{\, x \in \psi_f(K) \,|\, (T - \lambda id)^m x = 0 \quad (m >> 0)\,\}.$$

Since $\psi_f(K) \in D_c^b(\mathbb{C}_{f^{-1}(0)})$, we have a quasi-isomorphism $\bigoplus_{\lambda\in\mathbb{C}^\times} \psi_{f,\lambda}(K) \underset{qis}{\xrightarrow{\sim}} \psi_f(K)$. Similarly we have $\bigoplus_{\lambda\in\mathbb{C}^\times} \phi_{f,\lambda}(K) \underset{qis}{\xrightarrow{\sim}} \phi_f(K)$ as well. For convenience we set ${}^p\psi_f(K) := \psi_f(K)[-1]$ and ${}^p\phi_f(K) := \phi_f(K)[-1]$.

Let Y be a smooth hypersurface of X defined by $t = 0$. When $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, we have quasi-isomorphisms

$$\mathrm{DR}(\psi^m_{t,\alpha}(\mathcal{M})) \underset{qis}{\xrightarrow{\sim}} {}^p\psi_{t,e^{2\pi i\alpha}}(\mathrm{DR}(\mathcal{M})) \quad (-1 \leq \alpha < 0)$$
$$\mathrm{DR}(\phi^m_{t,0}(\mathcal{M})) \underset{qis}{\xrightarrow{\sim}} {}^p\psi_{t,1}(\mathrm{DR}(\mathcal{M}))$$

(see [**SGA7**]). Hence we obtain;

THEOREM 5.1 [**Kv**], [**Ma**]. *For a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we have*

$$\mathrm{DR}(\mathrm{gr}^V_\alpha \mathcal{M}) \underset{qis}{\xrightarrow{\sim}} \begin{cases} {}^p\psi_{t,e^{2\pi i\alpha}}(\mathrm{DR}(\mathcal{M})) & (-1 \leq \alpha < 0) \\ {}^p\phi_{t,e^{2\pi i\alpha}}(\mathrm{DR}(\mathcal{M})) & (-1 < \alpha \leq 0). \end{cases}$$

Moreover under the above quasi-isomorphisms we have the following correspondences:

$$\exp(-2\pi i t D_t) \leftrightarrow T$$
$$D_t : \mathrm{gr}^V_{-1} \mathcal{M} \longrightarrow \mathrm{gr}^V_0 \mathcal{M} \leftrightarrow can : {}^p\psi_{t,1}(\mathcal{M}) \longrightarrow {}^p\phi_{t,1}(\mathcal{M})$$
$$\frac{[\exp(-2\pi i t D_t) - 1]}{tD_t} t : \mathrm{gr}^V_0 \mathcal{M} \longrightarrow \mathrm{gr}^V_{-1} \mathcal{M} \leftrightarrow var : {}^p\phi_{t,1}(\mathcal{M}) \longrightarrow {}^p\psi_{t,1}(\mathcal{M}).$$

COROLLARY 5.2. *For a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we have*

$$^p\psi_t(\mathrm{D}_X(\mathrm{DR}(\mathcal{M}))) \xrightarrow[qis]{\sim} \mathrm{D}_Y\, {}^p\psi_t(\mathrm{DR}(\mathcal{M}))$$

$$^p\phi_t(\mathrm{D}_X(\mathrm{DR}(\mathcal{M}))) \xrightarrow[qis]{\sim} \mathrm{D}_Y\, {}^p\phi_t(\mathrm{DR}(\mathcal{M})).$$

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